Mutually penetrating motion of self-organized two-dimensional patterns of solitonlike structures

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Results of numerical simulations of a recently derived most general dissipative-dispersive partial differential equation describing evolution of a film flowing down an inclined plane are presented. They indicate that a novel complex type of spatiotemporal patterns can exist for strange attractors of nonequilibrium systems. It is suggested that real-life experiments satisfying the validity conditions of the theory are possible: the required sufficiently viscous liquids are readily available. [S1063-651X(97)00401-7]

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The phenomenon of pattern formation in nonequilibrium dissipative systems is currently a topic of active experimental and theoretical research (see, e.g., [1] for a recent progress review). Here we report our theoretical studies and numerical simulations of a two-dimensional (2D) evolution partial differential equation (PDE) approximating a flow down an inclined plane; it exhibits self-organization of a remarkably complex spatiotemporal pattern which then persists indefinitely in this dissipative-dispersive system. In certain cases discussed below, such a pattern consists of two *subpatterns* of two-dimensionally localized surface structures. One of these subpatterns is an essentially 1D arrangement of larger-amplitude *bulges* on the film surface which are nearly equidistantly aligned on (a number of) straight-line segments; those are surrounded by smaller-amplitude *bumps*, which constitute the second, latticelike subpattern filling up essentially the entire flow domain. Each of the two subpatterns moves as a whole; their velocities are different, and every elementary structure (a bulge as well as a bump) periodically collides with those of the other kind. In the collision of a bump with a bulge (or with a pair of neighboring bulges), the two structures pass through each other similarly to the well-known 1D Korteweg–de Vries solitons, returning to their precollisional shapes and speeds after the interaction.

Studies of wavy film flows on solid surfaces (the ''Kapitza problem'') have a considerable history. However, the nonlinear dynamics of wavy films is far from being fully understood (see, e.g., [2]; see [3,4] for recent progress reviews). Fortunately, the nonlinear coupled-PDE Navier-Stokes (NS) problem, additionally complicated with a free boundary, can be reduced to simpler *approximate* descriptions of the wave dynamics for certain domains of the parameter space. In the most favorable cases, such a description reduces to a *single* partial differential equation governing the evolution of film thickness. Recently, we (see [3]) have derived the most general evolution equation (EE) capable of all-time-valid description of a wavy liquid film (of a constant density ρ, kinematic viscosity ν, surface tension σ, and average thickness h_0) flowing down an inclined plane. Its dimensionless form is

$$\eta_t+4\eta\eta_z+\tfrac{2}{3}\delta\eta_{zz}-\tfrac{2}{3}\cot\theta\,\eta_{yy}+\tfrac{2}{3}W\nabla^4\eta+2\nabla^2\eta_z=0. \tag{1}$$

Here η is the deviation of film thickness from its average value of 1, z and y are the streamwise and spanwise coordinates, and $\nabla^2=\partial^2/\partial z^2+\partial^2/\partial y^2$. We have defined $\delta\equiv(4R/5-\cot\theta)$, where θ is the angle of inclination of the plane and $R=h_0U/\nu$ is the Reynolds number. Here $U=gh_0^2\sin\theta/(2\nu)$ where g is the gravity acceleration, and in Eq. (1) $W=\sigma/(2\rho\nu U)$ is the Weber number. All the dimensionless variables are measured in units based on h_0, U, and ρ. Equation (1) describes the film evolution in a reference frame moving with the velocity $2U$ in the streamwise direction.

In this paper we limit ourselves to the case of $\theta=\pi/2$, i.e., flow down a vertical wall. Then, we can transform the EE to a ''canonical'' form which will contain only *one* control parameter—by rescaling $\eta=N\tilde{\eta}$, $z=L\tilde{z}$, $y=L\tilde{y}$, and $t=T\tilde{t}$, where $N=2R/(5W)$, $L=\sqrt{5W/(4R)}$, and $T=(5^{3/2}/16)(W/R)^{3/2}$. Dropping the tildes in the notations of variables, the resulting canonical form of the EE is

$$\eta_t+\eta\eta_z+\nabla^2\eta_z+\epsilon(\eta_{zz}+\nabla^4\eta)=0. \tag{2}$$

The control parameter in this equation is

$$\epsilon=(1/3)\sqrt{4WR/5}. \tag{3}$$

Equations (1) and (2) have been *derived* directly from the fundamental NS equations by using an iterative procedure which is a variation of the so-called multiparameter perturbation approach (see, e.g., [5,3], and references therein; an earlier, more limited application of multiple independent perturbation parameters appears, e.g., in [6]). In addition to its leading to the most general EE, another advantage of this technique is that it yields the least restrictive conditions of theory validity. For the present case, they require that the following two dimensionless parameters be *independently* small:

$$\alpha\equiv\sqrt{R/W}\ll 1 \quad\text{and}\quad \beta\equiv\sqrt{R^3/W}\ll 1. \tag{4}$$

From the linear stability theory, the third (third-order derivative) term of Eq. (2) is purely dispersive, while all other linear terms are dissipative. Different limiting cases of Eq. (2) reproduce some known nonlinear equations, such as the 2D version of the Korteweg–de Vries (KdV) equation for $\epsilon\to 0$ and the 2D version of the Kuramoto-Sivashinsky (KS) equation for $\epsilon\to\infty$ (see also [7]). The 1D limit ($\partial_y=0$) of

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Eq. (2) is essentially the well-studied (see, e.g., references in [8]) equation introduced by Kawahara [9].

To exhibit interesting spatial behavior, a system should be sufficiently ''large.'' For the periodicity domain $0<y\leqslant 2\pi p$, $0<z\leqslant 2\pi q$ in our simulations of Eq. (2), we chose $5\leqslant p\leqslant 16$ and $16\leqslant q\leqslant 80$. We used spatial grids of up to $32p\times 32q$ nodes, with the Fourier pseudospectral method for spatial derivatives and with appropriate dealiasing. Time marching was done (in the Fourier space) by using Adams-Bashforth and/or Runge-Kutta methods. We checked the results by refining the space grids and time steps; by verifying the volume conservation, $\int \eta dy dz=0$; etc. A typical simulation run took $\sim 10^6$ time steps.

The initial values of η were chosen independently at each node from the interval $[-0.05,0.05]$ with uniform probability distribution. Due to the dissipativity of Eq. (2), the system evolves to an attractor, and so essentially ''forgets'' the initial conditions. For large values of the ''dissipativity'' parameter $\epsilon\gg 1$, as soon as the flow approaches its asymptotic state, the observed film surface is irregular in space and time; no spatial patterns are evident. The chaotic character of the attractor is indicated by the *positive* largest Liapunov exponent [which we found, similar to Deissler [10], by numerically integrating, along with Eq. (2), the linear equation that governs small disturbances of the solution of Eq. (2)]. This is in accordance with the fact that in the limit $\epsilon\rightarrow\infty$, Eq. (2) reduces to a 2D generalization of the Kuramoto-Sivashinsky equation, whose solutions on extended spatial domains are known to exhibit chaotic attractors. Regarding the transient behavior on the way to the attractor, our simulations of Eq. (2) with small ϵ corresponding to the parameter values of the experiment [2] have shown agreement [3] with their *transient* ''3D'' patterns and pattern transitions, including checkerboard patterns, synchronous instabilities, and solitary waves.

The main focus of the present paper is the presence of highly nontrivial orderly patterns in time-asymptotic states for the *strongly dispersive* cases, $\epsilon\ll 1$. Figure 1 shows snapshots of the film surface at large times for three different sets of parameter values. (The fact that by those times the systems have approached their asymptotic states is clear, e.g., from the corresponding plots of the evolution of ''energy'' $\int \eta^2 dy dz$ [see Fig. 2 corresponding to Fig. 1(a)]. We will speak of such numerically identified time-asymptotic states as *attractors,* although one needs to be cautious here: it is known that such extended systems may sometimes exhibit long transients. We find the largest Liapunov exponent to be positive in this case as well, suggesting a *strange* attractor.)

There are two subpatterns in Fig. 1(a): The V-shaped formation consisting of 13 large-amplitude bulges aligned into two straight lines moves as a whole downstream with a certain velocity, and the small-amplitude latticelike subpattern of bumps moves uniformly as well, but in the opposite (in our reference frame) direction. [Similar segregation of coherent structures into two subpatterns of different amplitudes is also seen for the nonsquare, large-aspect-ratio domains, Figs. 1(b) and 1(c).] This collision-course movement is evident in the cross-sectional space-time portrait shown in Fig. 3. Even though each bump changes its shape in irregular manner, the bump maintains its identity. In particular, the bumps do not seem to coalesce or break up, and just weakly interact with one another. Also, the height of a bulge *irregularly fluctuates*, by an amount which is approximately equal to the amplitude of the bumps.

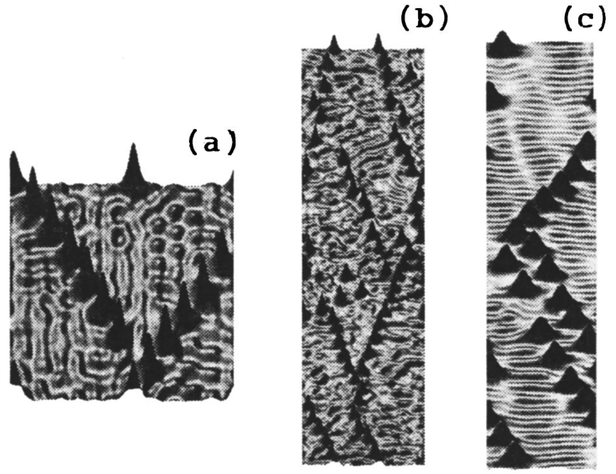

FIG. 1. Snapshots of the time-asymptotic film surface self-organized in simulations of Eq. (2), for three different cases (bulges move down the page here; for convenience of presentation, different axes may have different scales; in reality, all ''bulges'' and ''bumps'' have small slopes and are nearly axisymmetric). (a) $p=q=16$, $\epsilon^{-1}=50$, and $t=1.6\times 10^5$; (b) $(p,q)=(16,80)$, $\epsilon^{-1}=30$, and $t=5.98\times 10^4$; (c) $(p,q)=(5,60)$, $\epsilon^{-1}=25$, and $t=4.89\times 10^5$.

As a bump runs into a bulge, the bulge's amplitude increases momentarily, and then decreases again as a bump separates from the opposite side of the bulge (see Fig. 3). These interactions, unlike the irreversible coalescences of 1D pulses—discovered in [11] for a highly nonlinear dissipative equation—appear to be (almost) reversible, like the well-known interactions of 1D KdV solitons.

We note that *bulge* formations similar to that of Fig. 1(a) were discovered in [7] for $\epsilon^{-1}=25$ and $p=64/(2\pi)$ [they postulated an equation of the form (2) based on an equation of the form (1) derived in Ref. [12] for a partial case of an inclined film; in fact, that derivation was *not* valid for the *vertical* film]. But the authors of [7] seem to have overlooked (perhaps, because of inadequacy of the graphics tools they

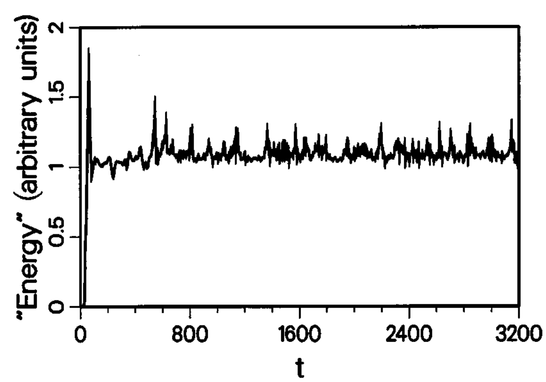

FIG. 2. Evolution of the surface deviation ''energy'' $\int \eta^2 dy dz$ from an initial small-amplitude ''white-noise'' surface to an attractor of Eq. (2). The snapshot Fig. 1(a) was taken near the end of this run. Note that the time unit here is 50 times that of Eq. (2).

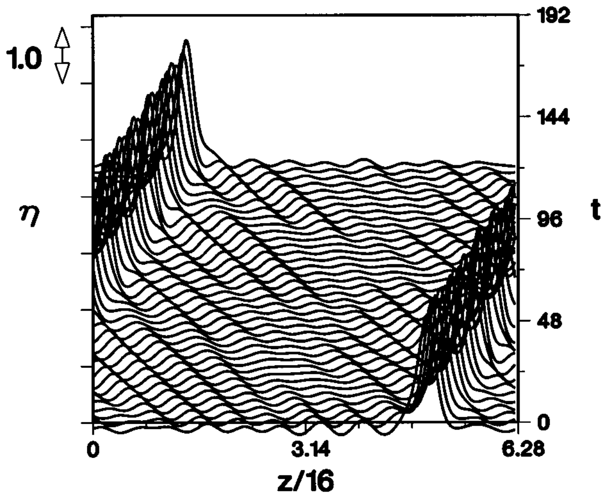

FIG. 3. Time sequence of instantaneous surface profiles in a fixed vertical cross section normal to the film (for $p=q=16$, $\epsilon^{-1}=50$; the time shown as 0 is in fact 1.6×10^5 counting from the start of the run). In particular, it is evident that the (large-amplitude) bulges move in one direction and (small-amplitude) bumps in the opposite direction.

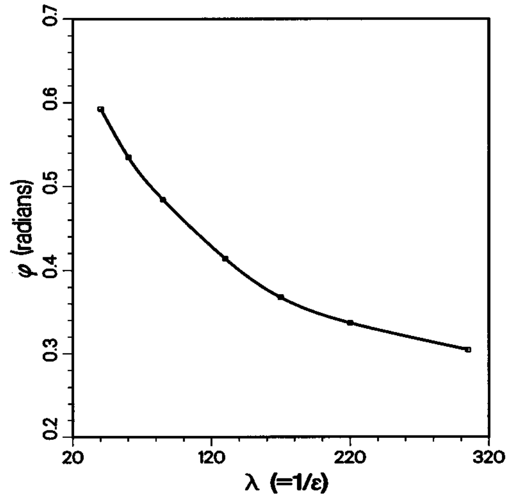

FIG. 4. Angle φ between (each) line of bulges and the streamwise direction varies with ϵ ($p=q=16$).

used) the second, bump subpattern—and thus the entire complex, dynamical character of the two-component order.

It is natural to inquire as to how the various quantities of the pattern scale with ϵ. We varied ϵ^{-1} between 25 and 305 for $p=q=16$. In one set of simulations, ϵ^{-1} was gradually decreased from 50 in relatively small steps of 5 (to allow the system to ''adiabatically'' adjust to the new parameter value), up to $\epsilon^{-1}=25$—at which point the line formations of bulges break down. In another set of simulations, ϵ^{-1} was increased from 50 in steps of 10 or 15 up to $\epsilon^{-1}=305$. In all cases, we find that the characteristic width of the bulge as well as the bump is of the order of ($\sim$) 1 independent of ϵ. The amplitude of bulges is also constant, ~ 1, as are the velocity of bulges and that of bumps. Only the amplitude of bumps changes; it scales as $\sim\epsilon$.

The V-shaped formation of bulges retains its form when ϵ is changed from 1/30 to 1/305. However, the (absolute value of the) angle φ of each bulge line with the streamwise axis decreases with ϵ, probably approaching some asymptotic value in the limit $\epsilon\to0$ (see Fig. 4; since there are no parameters remaining in this limit, the asymptotic angle should be just 0). It might be possible to explain this dependence by a theory of pairwise interaction of bulges through their (nonaxisymmetric) ''tails'' (similar to the theory [13] for 2D chemical-wave spirals).

When $\epsilon\ll1$, the dissipative terms in Eq. (2) can be treated as perturbations $\sim\epsilon$ of the 2D KdV equation

$$\eta_t+\eta\eta_z+\nabla^2\eta_z=0. \tag{5}$$

This equation does not seem to have any analytical solutions. However, by transforming to a reference frame moving with a velocity $c>0$ [replace η_t with $(-c\eta_z)$ in the equation], Petviashvili and Yan'kov [14] numerically obtained a stationary axially symmetric solitary-wave solution. By balancing the first term with the nonlinear term, $c\eta_z\sim\eta\eta_z$, and the latter term with the dispersive term, the characteristic amplitude and velocity of these solutions are found to be $\eta\sim c$ and $c\sim 1/L_s^2$ where L_s is the characteristic length scale, which is not uniquely determined by the KdV equation (5). However, the two dissipative terms of Eq. (2) will change L_s on a slow time scale, until they balance each other (this essential role of the small dissipative terms was revealed in Ref. [9] for the 1D case). This selects the soliton of $L_s\sim1$, which results in $c\sim1$ and $\eta\sim1$ as well, independent of ϵ. These estimates are clearly consistent with the numerical results for bulges reported above.

Motivated by the discovery of the second, small-amplitude subpattern, we examined the possibility of a corresponding second traveling-wave solution. If we transfer to the frame moving with a *negative* velocity $c=-a^2$, where a is a (real) constant, there are such solutions—with the nonlinear term being as small as the dissipative ones. Indeed, the leading-order equation then is $\nabla^2\eta_z+a^2\eta_z=0$, which is the well-known Helmholtz equation for η_z. There are solutions $\propto\sin Jy\sin Kz$ $(J^2+K^2=a^2)$. The balance between the (small) dissipative terms again determines $K\sim c\sim1$, and the balance of the dissipative terms with the nonlinear term yields $\eta\sim\epsilon$. We see that these length scale, amplitude, and velocity (including its sign) agree with those observed for the bumps in the numerical experiments as described above. Note that our assumption of the *negative* velocity is essential: with a positive velocity, one arrives at the *modified* Helmholtz equation, which does not have any oscillating solutions. There are only *exponential* solutions, which are unsuitable here. [We note that the Helmholtz equation has axially symmetric solutions as well, $\propto J_0(ar)$ where J_0 is the Bessel function (r is the radial coordinate). This solution is only weakly localized: it decays at spatial infinity as a power rather than exponentially. There is no such localized solution in the 1D case, $\partial_y=0$.]

One would naturally like to find some known types of patterns to which those reported here can be compared. There are several known cases (see, e.g., Ref. [15], and references therein) of indefinitely long *coexistence* of different types of patterns. However, in those cases each of the coexisting patterns is confined to its own spatial region: its constituent structures do not penetrate inside any ''alien'' pattern region. In contrast, we have seen that the bumps constantly pass through the region of bulges. Another possibility would be to look at the bulges and bumps as two traveling waves. However, in contrast to usual cases, the bulge ''wave'' is confined to an essentially 1D region, and there is a constant nonlinear interaction with the wave of bumps.

Similar to Eq. (2), we have derived an equation for a film flowing down a vertical *cylinder* (see Ref. [3] and references therein). In particular, one can see that if the (dimensionless) radius b of the cylinder is not too small ($b \gg \beta^{-1}$), the flow is well approximated by the planar-film equation (1). (With periodic boundary condition in the azimuthal direction; we note that this also justifies our use of *spanwise*-periodic BCs in the numerical simulations. As to the streamwise BC, we believe the solution becomes essentially insensitive to their specific type in the limit of large aspect ratio q/p, as, e.g., in Fig. 1. It would be interesting to check this with spatial-evolution simulations, such as those already conducted [16] for a different EE that coincides with the nondispersive limit of the above-mentioned EE [3].) One finds that with $\bar{h}_0 \sim 1$ mm, the cylinder (dimensional) radius $\bar{b} \sim 1$ cm, and under parametric conditions $\alpha \ll 1$, $\beta \ll 1$, and $\epsilon \ll 1$, for the waves (evolving as they propagate from the entrance end of not-too-long a cylinder to its exit end) to have enough time for approaching the attractor stage, the liquid should be several hundred times as viscous as water. For example, it could be glycerin with an admixture of water.

As a general conclusion, numerical 2D simulations of a realistic evolution PDE signal that nonequilibrium dissipative systems can spontaneously form *non*periodic, but nevertheless highly ordered spatial patterns (of compactly localized, solitonlike structures) which are of a remarkable complexity. In particular, the novel patterns consist of *subpatterns*—each of a different amplitude and each moving as a whole with its own velocity, *penetrating* through one another. Thus these subpatterns coexist as the constituent components of the overall—microscopically *nonsteady* but macroscopically permanent—self-organized spatiotemporal order characteristic of the system motion on such an attractor.

The particular dissipative-dispersive PDEs (1) and (2) have been consistently derived from the full Navier-Stokes problem to provide a controllably close approximation to the evolution of a liquid film flowing down a vertical plane. The unconventional perturbative approach used in this derivation has the advantage of yielding the least restrictive conditions of the validity of the theory. To satisfy those validity conditions for a possible (terrestrial) experiment designed to observe patterns of the novel type on a film flowing down a vertical cylinder, the film liquid should be much more viscous than water; fortunately, suitable liquids are readily available.

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